Anisotropic extension of the Brans-Dicke gravity

Taeyoon Moon[a1] and Phillial Oh[b2]

[a] *Center for Quantum Space-time, Sogang University, Seoul, 121-742, Korea*

[b] *Department of Physics and Institute of Basic Science, Sungkyunkwan University, Suwon 440-746 Korea*

Abstract

We consider the ADM formalism of the Brans-Dicke theory and propose an anisotropic extension of the theory by introducing five free parameters. We find that the resulting theory reveals many interesting aspects which are not present in the original BD theory. We first discuss the ghost instability and strong coupling problems which are present in the gravity theory without the full diffeomorphism symmetry and show that they can be avoided in a region of the parameter space. We also perform the post-Newtonian approximation and show that the constraint of the Brans-Dicke parameter $\omega_{\rm BD}$ being large to be consistent with the solar system observations could be evaded in the extended theory. We also discuss that accelerating Universe can be achieved without the need of the potential for the Brans-Dicke scalar.

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[1]e-mail address: dpproject@skku.edu
[2]e-mail address: ploh@skku.edu

1 Introduction

There are many alternative theories and extensions of the Einstein's general relativity. Motivations to consider them are diverse. They range from a classical straightforward extension of general relativity like the scalar-tensor theory to quantum gravity such as string theory. A comprehensive review can be found in a recent article [1]. The classical modification is based on the observational rationale that the predictions of the general relativity has been successful only on the solar system scale and has never been tested on the cosmological scale. Recently, it has another input from cosmology, that is, to search for the theoretical foundation of the accelerating Universe [2] has become one of the most fundamental problems in modern cosmology. There are two main avenues. The first one is to assume an unknown source of energy which is repulsive in nature referred to as the dark energy and various proposals for its origin have been put forward [3]. The other is to consider an alternative theory of the Einstein's theory of general relativity and modify the gravity in the IR limit. We are still awaiting more accurate observational data to distinguish between them. Therefore modification of general relativity is a very important subject and is attracting a great deal of interest with extensive research activities being reported (see [1] and references therein).

Among the many alternatives, scalar-tensor theory [4] is especially interesting. In this theory, the scalar field enters in a nontrivial manner, specifically through non-minimal coupling term and it is based on solid foundation of general relativity. Typical examples would be the Brans-Dicke (BD) theory [5] and the gravity with a dilaton field arising for instance in the string theory. The scalar field in this type of theory provides many distinctive theoretical aspects, for example like induced gravity [6] and explanation of the behavior of the universe in the early inflationary stage as well as the late stage [4] in the cosmological context.

Especially, we attempt to modify the BD theory which is known as the prototype of the scalar-tensor theory and is one of the simplest alternative to Einstein's theory of general relativity (GR). BD theory was firstly designed to properly incorporate the Mach's principle into General relativity by replacing the gravitational constant G with a scalar field ϕ, which can vary with space and time. It is well-known that the observational constraints on BD theory is restricted by the astronomical tests in the solar system, i.e., the BD parameter $\omega_{\rm BD} > 50000$, being obtained from the observable Post-Newtonian parameter $\gamma_{\rm obs} - 1 = (2.1 \pm 2.3) \times 10^{-5}$ [7, 8].

Modifications of Brans-Dicke theory have been actively pursued. Most notable is the introduction of the potential for the Brans-Dicke scalar. For example, in Ref. [9, 10], BD theory with a suitable self-interacting potential for the BD scalar field was proposed to account for the accelerated expanding of the Universe. In another example, it can be shown that $f(R)$ gravity in the metric formalism is equivalent to BD theory with a potential and the parameter $\omega_{\rm BD} = 0$ [11].

Another alternative is an anisotropic extension. It is motivated by the Horava-Lifshitz (HL) gravity [12] which was proposed as a quantum gravity theory. It is based on anisotropic

scaling of space and time as a fundamental symmetry, abandoning the Lorentz symmetry at short distance. In the IR limit, the theory reduces to GR when the Lorentz violating parameter λ becomes 1. There has been considerable interest in the theory and various aspects have been investigated [13]. Among others, a BD type of generalization of the Horava-Lifshitz gravity was shown to be possible within the detailed balance condition in Ref. [14], and the resulting theory reduces in the IR limit to the usual BD theory with a negative cosmological constant. From the theoretical point of view, breaking of the full diffeomorphism invariance coming from the anisotropy puts strong constraint in the resulting theory. The breaking causes the theory to have an additional scalar degree of freedom, whose behavior could create serious problems, such as a ghost or classical instability or the strong coupling problem. On the experimental side, it should pass the solar system test which requires $\omega_{\rm BD}$ has to be big in the original BD theory.

In this paper, we propose a more straightforward modification of BD theory starting from the ADM formalism and discuss the above issues. A close look at the ADM formalism reveals that the original BD theory can be anisotropically extended by introducing four more free parameters. Depending on the values of these parameters, the theory shows many interesting aspects which are not present in the original BD theory. For example, the large value of $\omega_{\rm BD}$ in the original BD theory to be consistent with the solar system observations could be evaded in the modified theory. It can be also shown that the ghost instability and strong coupling problems which are present in the gravity theory without the full diffeomorphism symmetry can be avoided within some range of the free parameters. Another distinctive feature is that the accelerating Universe could be achieved without the help of some potential contrary to the original BD theory.

The paper is organized as follows. In Sec. 2, we consider anisotropic BD (aBD) action with five parameters which reduces to the ordinary (isotropic) BD action for particular choices of the four parameters. In Sec. 3, we consider a quadratic action through the perturbation analysis of the aBD gravity in order to investigate the pathological behaviors of scalar graviton mode,. In Sec. 4, we perform the perturbation analysis up to cubic order and check whether the strong coupling problem can be cured to this order. In Sec. 5 we study the observational constraint on the aBD theory and compare with the experimental results. The conclusion and discussions are given in Section 6.

2 Anisotropic Brans-Dicke action

In order to construct an anisotropic Brans-Dicke gravity with additional parameters, let us first consider an isotropic Brans-Dicke gravity in the ADM formalism, whose metric is parametrized by

$$ds^2_{\rm ADM} = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{2.1}$$

where N is the lapse function, N_i is the shift function, and g_{ij} is the three dimensional metric. For the ADM metric, Brans-Dicke action is given by [14]

$$\begin{aligned} S_{\rm BD} &= \int d^4x\sqrt{-g_{(4)}}\left\{\phi R_{(4)} - \frac{\omega_{\rm BD}}{\phi}g^{\mu\nu}_{(4)}\partial_\mu\phi\partial_\nu\phi\right\} \\ &= \int dtd^3xN\sqrt{g}\Bigg\{\phi(K_{ij}K^{ij}-K^2) - 2K\pi + \omega_{\rm BD}\phi^{-1}\pi^2 \\ &\qquad\qquad + \phi R - 2g^{ij}\nabla_i\nabla_j\phi - \omega_{\rm BD}\phi^{-1}g^{ij}\nabla_i\phi\nabla_j\phi\Bigg\}, \end{aligned} \tag{2.2}$$

where ϕ and $\omega_{\rm BD}$ is the Brans-Dicke scalar and Brans-Dicke parameter respectively. g_{ij} and R are the three dimensional metric tensor and the Ricci scalar. The extrinsic curvature K_{ij} and π take the forms

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \tag{2.3}$$

$$\pi = \frac{1}{N}(\dot{\phi} - \nabla_i\phi N^i), \tag{2.4}$$

where the dot denotes differentiation with respect to t. The first line of (2.2) is the kinetic (K) term and second line is the potential (V) term.

We first notice that the BD gravity (2.2) preserves the full diffeomorphism symmetry. We can extend it to aBD theory by introducing free parameters which explicitly break the diffeomorphism invariance and consider the most general action as follows [1]:

$$S_{\rm aBD} = S^{\rm K}_{\rm aBD} + S^{\rm V}_{\rm aBD}, \tag{2.5}$$

where

$$S^{\rm K}_{\rm aBD} = \int dtd^3xN\sqrt{g}\ \left\{\phi\left(K_{ij}K^{ij} - \lambda K^2\right) - 2\eta_1 K\pi + \omega_1\phi^{-1}\pi^2\right\}, \tag{2.6}$$

$$S^{\rm V}_{\rm aBD} = \int dtd^3xN\sqrt{g}\ \left(\phi R - 2\eta_2\nabla_i\nabla^i\phi - \omega_2\phi^{-1}\nabla_i\phi\nabla^i\phi\right). \tag{2.7}$$

In this action, the parameters $\lambda,\ \eta_{1,2},\ \omega_{1,2}$ are dimensionless constants. Note that when $\lambda \neq 1$, $\eta_1 \neq 1$, $\eta_2 \neq 1$, and $\omega_1 \neq \omega_2$, the action (2.5) does not have the full diffeomorphism invariance, but is invariant under the foliation-preserving diffeomorphism:

$$\begin{aligned} \delta x^i &= -\zeta^i(t,\mathbf{x}),\ \delta t = -f(t), \\ \delta g_{ij} &= \partial_i\zeta^k g_{jk} + \partial_j\zeta^k g_{ik} + \zeta^k\partial_k g_{ij} + f\dot{g}_{ij}, \\ \delta N_i &= \partial_i\zeta^j N_j + \zeta^j\partial_j N_i + \dot{\zeta}^j g_{ij} + f\dot{N}_i + \dot{f}N_i, \\ \delta N &= \zeta^j\partial_j N + f\dot{N} + \dot{f}N, \\ \delta\phi &= \zeta^k\partial_k\phi + f\dot{\phi}. \end{aligned} \tag{2.8}$$

[1]This extension is based on the fact that each term of $2K\pi$, $2g^{ij}\nabla_i\nabla_j\phi$, $\phi^{-1}\pi^2$, and $\phi^{-1}g^{ij}\nabla_i\phi\nabla_j\phi$ in the action (2.2) is invariant under foliation-preserving diffeomorphism (2.8).

We point out that the parameter $\eta_1(\ \eta_2)$ is associated with deviation from the BD theory along the normal (spatial) direction of the leaves of the foliation. Moreover, we introduced two BD parameters ω_1 and ω_2 which need not to be the same in the anisotropic case. With the choice of the parameters of $\lambda = 1,\ \eta_1 = \eta_2 = 1$, and $\omega_1 = \omega_2 = \omega_{\rm BD}$, the action (2.5) reduces to (isotropic) Brans-Dicke action (2.2). For a constant aBD scalar field, it becomes Lorentz violating Einstein-Hilbert action with an anisotropic parameter λ. Note also that the aBD action (2.5) is still invariant with respect to global (isotropic) conformal transformation

$$N \to \Omega N, \quad N_i \to \Omega^2 N_i, \quad g_{ij} \to \Omega^2 g_{ij}, \quad \phi \to \Omega^{-2}\phi, \tag{2.9}$$

where Ω is an arbitrary constant. We further remark that for $\Omega = \Omega(t, \mathbf{x})$, the conformal symmetry under the transformation (2.9) can be extended to local conformal symmetry if one chooses

$$\begin{aligned} \eta_1 &= -\frac{1}{2}(1-3\lambda), \quad \omega_1 = \frac{3}{4}(1-3\lambda), \\ \eta_2 &= 1, \quad \omega_2 = -\frac{3}{2}, \end{aligned} \tag{2.10}$$

which implies that for $\lambda = 1$ case, i.e., $\eta_1 = \eta_2 = 1,\ \omega_1 = \omega_2 = -3/2$, the aBD action (2.5) reduces to a conformally invariant isotropic action [15].

3 Scalar graviton mode in the quadratic action

In order to investigate the behavior of graviton mode in the Minkowski background ($g_{ij} = \delta_{ij},\ \phi = \phi_0$), we first consider the following scalar perturbations of the metric and aBD scalar field for the Minkowski background up to the linear order

$$N = e^{\alpha}, \quad N_i = \partial_i \beta, \quad g_{ij} = e^{-2\psi}\delta_{ij}, \quad \phi = \phi_0 + \varphi. \tag{3.1}$$

In these perturbations, $\alpha,\ \beta,\ \psi$, and φ are functions of space and time. Substituting the above perturbations into the action (2.5), one finds the quadratic action ($\Box \equiv \partial_i\partial^i$)

$$\begin{aligned} S^{(2)} = \int dt d^3x \Big\{ & 3\phi_0(1-3\lambda)\dot{\psi}^2 + 2\phi_0(1-3\lambda)\dot{\psi}\Box\beta + \phi_0(1-\lambda)(\Box\beta)^2 + 2\eta_1(3\dot{\varphi}\dot{\psi} + \dot{\varphi}\Box\beta) \\ & + \omega_1\phi_0^{-1}\dot{\varphi}^2 + 4\varphi\Box\psi + 4\phi_0\alpha\Box\psi - 2\phi_0\psi\Box\psi - 2\eta_2\alpha\Box\varphi - \omega_2\phi_0^{-1}(\partial_i\varphi)^2 \Big\}. \end{aligned} \tag{3.2}$$

Variation of the fields α and β of the quadratic action leads to the (local) Hamiltonian and momentum constraints

$$\varphi - \frac{2}{\eta_2}\phi_0\psi = 0, \tag{3.3}$$

$$\Box\beta + \frac{1}{1-\lambda}\left(1 - 3\lambda + \frac{2\eta_1}{\eta_2}\right)\dot{\psi} = 0, \tag{3.4}$$

respectively.

From the constraints (3.3) and (3.4), we can replace the fields φ and $\Box\beta$ with ψ. After taking the integration by parts, the quadratic action (3.2) can be rewritten as

$$S^{(2)} = 2\int dtd^3x \left\{ -\frac{1}{c_\psi^2}\dot{\psi}^2 + \left(\frac{2\omega_2}{\eta_2^2} + \frac{4}{\eta_2} - 1\right)\phi_0\psi\Box\psi \right\}, \tag{3.5}$$

where

$$c_\psi^2 = \frac{1-\lambda}{\phi_0}\left\{3\lambda - 1 + \frac{2\eta_1^2}{\eta_2^2} - \frac{4\eta_1}{\eta_2} + \frac{2(\lambda-1)\omega_1}{\eta_2^2}\right\}^{-1}. \tag{3.6}$$

We notice two things. First is that the perturbation φ can be completely eliminated due to the Hamiltonian constraint (3.3) in favor of the scalar graviton mode. The other is that in the limit[2] $\eta_2 \to \infty$, we find the action is exactly the same with the quadratic action in the HL gravity (for $\phi_0 = 1$)

$$S_{\rm HL}^{(2)} = 2\int dtd^3x \left\{ -\frac{1}{c_{\rm HL}^2}\dot{\psi}^2 - \psi\Box\psi \right\}, \tag{3.7}$$

with $c_{\rm HL}^2 = (1-\lambda)/(3\lambda-1)$. It can be easily checked that in the quadratic action (3.7), the scalar graviton ψ has serious problems, i.e., classical instability in the case of $c_{\rm HL}^2 < 0$ or the presence of ghost when $c_{\rm HL}^2 > 0$. Interestingly, in our framework such pathological behaviors do not show up with

$$c_\psi^2 < 0, \quad 0 < \frac{2\omega_2}{\eta_2^2} + \frac{4}{\eta_2} - 1, \tag{3.8}$$

which can be satisfied with a wide range of the parameters region.

Let us focus on the special cases $\eta_1 = \eta_2$ (otherwise noticed) which have rather special consequences. For $\eta_1 = \eta_2 \equiv \eta$, the quadratic action (3.5) becomes

$$S^{(2)} = 2\phi_0\int dtd^3x \left\{ \left(3 + \frac{2\omega_1}{\eta^2}\right)\dot{\psi}^2 + \left(\frac{2\omega_2}{\eta^2} + \frac{4}{\eta} - 1\right)\psi\Box\psi \right\}, \tag{3.9}$$

and it shows that the dependency on the parameter λ for the scalar graviton completely disappears. This also implies the quadratic action (3.5) with (3.6) is devoid of the singularity associated with the limit $\lambda \to 1$. The condition (3.8) of avoiding the classical instability and ghost mode in this case is given by

$$\omega_1 > -\frac{3}{2}\eta^2, \quad \omega_2 > -2 + \frac{1}{2}(\eta-2)^2. \tag{3.10}$$

[2]This limit of $\eta_2 \to \infty$ is reminiscent of $\alpha \to \infty$ which is the coefficient of $(\nabla_i N/N)^2$ term introduced in Ref. [16] (see also Ref. [17] for details).

Note that ω_2 always has to be greater than -2 for any value of η. For $\eta = 1$, $\omega_1 > -\frac{3}{2}$ and $\omega_2 > -\frac{3}{2}$, which coincides the ghost free case of the isotropic BD theory [18], $\eta_1 = \eta_2 = 1$, $\omega_1 = \omega_2 = \omega_{\rm BD} > -\frac{3}{2}$. Another case of interest is when $\eta = 2$. In this case, we have $\omega_1 > -6$, $\omega_2 > -2$. We will see that for this particular value, ω_2 does not have to be a large value in order to pass the solar test as in the original BD theory [see Eq. (5.7)]. As η becomes larger, ω_1 can be more negative, whereas ω_2 shifts away from the value -2 in the positive direction.

4 Strong coupling in the cubic action

In the previous section, we investigated the behavior of graviton mode in the quadratic action and showed that the pathological behavior can be cured in a wide range of parameters with the condition of Eq. (3.8) being satisfied. Another important task is to examine if the theory gets strongly coupled at the cubic action level in the limit of $\lambda \to 1$ [19, 20].

In order to study with the cubic order interaction terms [21], we first recall the non-linear scalar perturbations (3.1) around the Minkowski background and substitute this into the action (2.5). After some tedious manipulations one can find that the cubic-order action is given by

$$
\begin{aligned}
S^{(3)} \;=\;& \\
&\int dtd^3x\bigg\{-2\varphi(\partial\psi)^2+2\phi_0\psi(\partial\psi)^2-2\alpha\phi_0(\partial\psi)^2-4\varphi\psi\Box\psi+4\alpha\varphi\Box\psi+2\phi_0\alpha^2\Box\psi \\
&+2\phi_0\psi^2\Box\psi-4\phi_0\alpha\psi\Box\psi-\eta_2\alpha^2\Box\varphi-\eta_2\psi^2\Box\varphi+2\eta_2\alpha\psi\Box\varphi-2\eta_2\psi\partial_i\varphi\partial_i\psi \\
&+2\eta_2\alpha\partial_i\varphi\partial_i\psi+\omega_2\phi_0^{-1}\psi(\partial\varphi)^2-\omega_2\phi_0^{-1}\alpha(\partial\varphi)^2+\omega_2\phi_0^{-2}\varphi(\partial\varphi)^2-9(1-3\lambda)\phi_0\psi\dot{\psi}^2 \\
&-2(1-3\lambda)\phi_0\psi\dot{\psi}\Box\beta-2(1-3\lambda)\phi_0\dot{\psi}\partial_i\psi\partial_i\beta-2(1-\lambda)\phi_0\Box\beta\partial_i\psi\partial_i\beta+\phi_0\psi(\partial_i\partial_j\beta)^2 \\
&+4\phi_0\partial_i\partial_j\beta\partial_i\beta\partial_j\psi-\lambda\phi_0\psi(\Box\beta)^2-3(1-3\lambda)\phi_0\alpha\dot{\psi}^2-2(1-3\lambda)\phi_0\alpha\dot{\psi}\Box\beta \\
&-\alpha\phi_0(\partial_i\partial_j\beta)^2+\phi_0\lambda\alpha(\Box\beta)^2+3(1-3\lambda)\varphi\dot{\psi}^2+2(1-3\lambda)\varphi\dot{\psi}\Box\beta+\varphi(\partial_i\partial_j\beta)^2 \\
&-\lambda\varphi(\Box\beta)^2-2\eta_1\Big(9\psi\dot{\psi}\dot{\varphi}+3\alpha\dot{\psi}\dot{\varphi}+3\dot{\psi}\partial_i\varphi\partial_i\beta+\dot{\varphi}\partial_i\beta\partial_i\psi+\alpha\dot{\varphi}\Box\beta+\psi\dot{\varphi}\Box\beta \\
&+\partial_i\varphi\partial_i\beta\Box\beta\Big)-\omega_1\Big(\phi_0^{-1}\alpha\dot{\varphi}^2+3\phi_0^{-1}\psi\dot{\varphi}^2+\phi_0^{-2}\varphi\dot{\varphi}^2+2\phi_0^{-1}\dot{\varphi}\partial_i\varphi\partial_i\beta\Big)\bigg\}. \qquad (4.1)
\end{aligned}
$$

Using the first-order Hamiltonian and momentum constraints (3.3), (3.4) obtained in the previous section, the above action (4.1) reduces to

$$
S^{(3)} = \phi_0\int dtd^3x\left\{A_1\psi(\partial_i\psi)^2+A_2\dot{\psi}\partial_i\psi\partial_i\left(\frac{\dot{\psi}}{\Box}\right)+A_3\psi\left(\frac{\partial_i\partial_j}{\Box}\dot{\psi}\right)^2+A_4\psi\dot{\psi}^2\right\}. \qquad (4.2)
$$

where

$$
\begin{aligned}
A_1 &= \frac{8\omega_2}{\eta_2^3} + \frac{4\omega_2}{\eta_2^2} + \frac{12}{\eta_2} - 2 \\
A_2 &= -\frac{8\eta_1 + 4(1-3\lambda)\eta_2}{(-1+\lambda)^2\eta_2^3}\Big\{2\eta_1^2 - 4\eta_1\eta_2 + (3\lambda-1)\eta_2^2 + 2(\lambda-1)\omega_1\Big\} \\
A_3 &= -\frac{3\eta_2 - 2}{(-1+\lambda)^2\eta_2^3}\Big\{2\eta_1 + (1-3\lambda)\eta_2\Big\}^2 \\
A_4 &= \frac{1}{(-1+\lambda)^2\eta_2^3}\Big\{(1+\lambda)(3\lambda-1)(3\eta_2-2)\eta_2^2 - 4\eta_1^2\Big(2\lambda + 3\eta_2(\lambda-2)\Big) \\
&\qquad -4\eta_1\eta_2\Big(2 - 6\lambda + 3\eta_2(1+\lambda)\Big) - 4\omega_1(\lambda-1)^2(3\eta_2+2)\Big\}.
\end{aligned}
$$

Note that when $\eta_2 \to \infty$ the above action (4.2) once again reduces to the cubic action in the HL gravity as

$$
S^{(3)}_{\rm HL} = 2\int dtd^3x\Bigg\{ -\psi(\partial_i\psi)^2 + \frac{2}{c^4_{\rm HL}}\dot\psi\partial_i\psi\partial_i\left(\frac{\dot\psi}{\Box}\right) + \frac{3}{2}\left[-\frac{1}{c^4_{\rm HL}}\psi\left(\frac{\partial_i\partial_j}{\Box}\dot\psi\right)^2 + \frac{2c^2_{\rm HL}+1}{c^4_{\rm HL}}\psi\dot\psi^2\right]\Bigg\}. \tag{4.3}
$$

It is well-known that the above action is confronted with the strong coupling problem in the limit $\lambda \to 1$ [19, 20]. On the other hand, in the case of $\eta_1 = \eta_2 \equiv \eta$ of our main focus, the above coefficients $A_2,\ A_3,\ A_4$ considerably simplify to become

$$
A_2 = 12\left(3 + \frac{2\omega_1}{\eta^2}\right), \quad A_3 = -9\left(3 - \frac{2}{\eta}\right), \quad A_4 = 9 - \frac{6}{\eta} - \frac{4\omega_1(2+3\eta)}{\eta^3},
$$

which again shows that there is no dependency on the parameter λ for the scalar graviton up to the cubic order. To see this more closely, we introduce the canonical variable $\hat\psi = \sqrt{2}\psi/|c_\psi|$ with $|c_\psi|^{-2} = \phi_0(3+2\omega_1/\eta^2)$, in terms of which the quadratic action (3.9) becomes

$$
S^{(2)} = \int dtd^3x\left\{\dot{\hat\psi}^2 + \left(\frac{2\omega_2}{\eta^2} + \frac{4}{\eta} - 1\right)\phi_0|c_\psi|^2\hat\psi\Box\hat\psi\right\}. \tag{4.4}
$$

By using the canonical variable $\hat\psi$, the cubic action (4.2) can be written as

$$
\begin{aligned}
S^{(3)} = \frac{1}{2\sqrt{2}}\int dtd^3x\Bigg\{&\left(\frac{8\omega_2}{\eta^3} + \frac{4\omega_2}{\eta^2} + \frac{12}{\eta} - 2\right)\phi_0|c_\psi|^3\hat\psi(\partial_i\hat\psi)^2 + 12|c_\psi|\dot{\hat\psi}\partial_i\hat\psi\partial_i\left(\frac{\dot{\hat\psi}}{\Box}\right) \\
-9\left(3 - \frac{2}{\eta}\right)\phi_0|c_\psi|^3\hat\psi\left(\frac{\partial_i\partial_j}{\Box}\dot{\hat\psi}\right)^2 &+ \left(\left(27 + \frac{6}{\eta}\right)\phi_0|c_\psi|^3 - 2\left(3 + \frac{2}{\eta}\right)|c_\psi|\right)\hat\psi\dot{\hat\psi}^2\Bigg\}.
\end{aligned} \tag{4.5}
$$

This action clearly shows non dependency on the parameter λ and that we do not need to consider any fine-tuning for the interactions to be regular.

Before closing this section, we comment on the case with local conformal invariance. It turns out that with the choice of the parameters as in (2.10), the time dependent cubic terms of ψ in the action (4.2) vanish and we get

$$S^{(3)} = \phi_0 \int dt d^3x \left\{-8\psi(\partial_i\psi)^2\right\}. \tag{4.6}$$

This shows that the strong coupling problem does not show up at the cubic-order perturbation of the action with the local conformal invariance.

5 Cosmological constraint on anisotropic BD theory

In this section we investigate the cosmological tests, which can provide the observational constraints for alternative theories of gravitation. For this purpose, we first consider the solar system test. Taking into account the static part of the perturbations in Sec.3, the Lagrangian (3.2) with a static point-like source term of mass M_s can be written by [22,23]

$$\mathcal{L}_{\text{static}} = \left\{4\varphi\Box\psi + 4\phi_0\alpha\Box\psi - 2\phi_0\psi\Box\psi - 2\eta_2\alpha\Box\varphi - \omega_2\phi_0^{-1}(\partial_i\varphi)^2\right\} - M_s\alpha\delta^3(\mathbf{x}). \tag{5.1}$$

From varying for the fields $\psi,\ \alpha$, and φ, we obtain the following equations:

$$\begin{aligned}\Box\varphi + \phi_0\Box\alpha - \phi_0\Box\psi &= 0\\ 4\phi_0\Box\psi - 2\eta_2\Box\varphi - M_s\delta^3(\mathbf{x}) &= 0\\ 2\Box\psi - \eta_2\Box\alpha + \omega_2\phi_0^{-1}\Box\varphi &= 0.\end{aligned} \tag{5.2}$$

The corresponding solutions to Eq. (5.2) are

$$\psi = \frac{\eta_2 + \omega_2}{2+\omega_2}\alpha, \qquad \varphi = \frac{(\eta_2 - 2)\phi_0}{\omega_2 + 2}\alpha, \tag{5.3}$$

where α is

$$\alpha = -\frac{(\omega_2+2)M_s}{8\pi\phi_0(2\omega_2 - \eta_2^2 + 4\eta_2)|\mathbf{x}|}. \tag{5.4}$$

It is well-known that the linear expansion of the metric (3.1) can be expressed by the Newtonian potential u and the Post-Newtonian parameter γ as follows:

$$g_{00} = -N^2 = -1 - 2\alpha = -1 + 2u, \tag{5.5}$$
$$g_{ij} = (1-2\psi)\delta_{ij} = (1+2\gamma u)\delta_{ij}, \tag{5.6}$$

which implies that for the solution (5.3) $u = -\alpha$ and the parameter γ can be written by

$$\gamma = \frac{\psi}{\alpha} = \frac{\eta_2 + \omega_2}{2 + \omega_2} = 1 + \frac{\eta_2 - 2}{\omega_2 + 2}. \tag{5.7}$$

Let us concentrate on the case $\eta_1 = \eta_2 = \eta$ again. The solar system tests currently constrain γ as [7]

$$|\gamma - 1| < 2 \times 10^{-5}. \tag{5.8}$$

For the case of the BD theory ($\eta = 1,\ \omega_2 = \omega_{\rm BD}$), we see that the above condition (5.8) together with (5.7) restricts the region of $\omega_{\rm BD}$ to

$$\omega_{\rm BD} > 50000. \tag{5.9}$$

Note, however, that unlike the BD theory, this large value constraint on $\omega_{\rm BD}$ (5.9) can be circumvented in the anisotropic BD case when η is very close to 2 as can be seen from

$$|\eta - 2| < 2 \times 10^{-5}(\omega_2 + 2) \tag{5.10}$$

obtained from Eq. (5.7). Especially, when $\eta = 2$, any $\omega_2 > -2$ is allowed which is also consistent with the stability and no-ghost condition of Eq. (3.10). In addition, for the solution (5.4) of α one can find the effective Newton's constant G_{eff} as

$$G_{eff} = \frac{\omega_2 + 2}{8\pi\phi_0(2\omega_2 - \eta^2 + 4\eta_2)}, \tag{5.11}$$

being obtained from the Newtonian potential, i.e., $u = G_{eff}M_s/|\mathbf{x}|$. In the General Relativity limit ($\omega_2 = \omega_{\rm BD} \to \infty$) we have $G_{eff} \to G_N$ for the substitution of $\phi_0 \leftrightarrow 1/(16\pi G_N)$ with the Newton's constant G_N. It is interesting to notice that in the anisotropic BD theory the reduction $G_{eff} \to G_N$ can be achieved alternatively with appropriate choices of the free parameters different from the BD theory. Especially, when choosing the $\eta = 2$ once again, G_{eff} reduces to the Newton's constant G_N irrespective of the value of ω_2.

On the other hand, we can also obtain the cosmological constraint from a ratio factor of G_c/G_{eff} (G_c being the effective cosmological gravitational constant defined in Eq. (5.12)) which is related to the primordial helium abundance [24]. Substituting the FRW metric ansatz and $\phi = \phi(t)$ into the action (2.5) with matter term included and varying the action with respect to the lapse function $N(t)$, we find the standard Friedmann equation can be written as

$$H^2 = \frac{8\pi G_c}{3}\rho, \tag{5.12}$$

where H is the Hubble parameter, ρ is total matter density of the Universe, and the effective cosmological gravitational constant G_c is given by

$$G_c = \frac{1}{8\pi\phi(3\lambda - 1)}. \tag{5.13}$$

It is pointed out that G_c is equivalent to G_N for the substitution of $\phi = \phi_0 \leftrightarrow 1/(16\pi G_N)$ in the limit of $\lambda \to 1$. In our case for $\phi = \phi_0$, we obtain the cosmological constraint from the observational bound [24] of G_c/G_{eff} as

$$\Big|\frac{G_c}{G_{eff}} - 1\Big| = \Big|\frac{3(\lambda-1)}{3\lambda-1} + \frac{(\eta-2)^2}{(3\lambda-1)(\omega_2+2)}\Big| < 0.125. \tag{5.14}$$

Comparing (5.10) with the bound (5.14) by using (3.10), we find that the allowed range of the parameter λ is

$$-\frac{2}{3} < \lambda - 1 < 0.095 \approx 10^{-1} \tag{5.15}$$

which imposes a rather loose constraint around $\lambda = 1$ [16].

6 Conclusion and discussion

In this paper, we constructed an anisotropic Brans-Dicke gravity, which includes five free parameters, i.e., $\lambda,\ \eta_1,\ \eta_2,\ \omega_1$, and ω_2. In the case of $\lambda = \eta_1 = \eta_2 = 1$ and $\omega_1 = \omega_2 = \omega_{\rm BD}$, the gravity reduces to the ordinary Brans-Dicke action with a BD parameter $\omega_{\rm BD}$. When fixing the aBD scalar field ϕ to be a constant value, it becomes Lorentz-violating Einstein-Hilbert action with an anisotropic parameter λ.

We found that in the perturbation around the Minkowski background and constant aBD scalar field the scalar graviton at the quadratic as well as cubic order in the aBD action does not show any pathological behaviors within some parameter range. This suggests that the aBD gravity can be a viable theoretical model. Especially, the case $\eta_1 = \eta_2$ reveals intriguing property of λ independence of both the quadratic and cubic actions. Also in the context of cosmological model, we have checked that unlike the case of the BD theory which is imposed by $\omega_{\rm BD} > 50000$ to be consistent with the experimental observations, this large value can be evaded in the aBD theory with the special value of $\eta = 2$ which is one of the novel feature of the aBD theory. But the origin of these special properties associated with $\eta = 2$ is a puzzling aspect which needs to be investigated further.

We conclude with a couple of comments on the issues related to the BD theory. First, it is well-known that in the standard BD model without potential, there is no accelerated expansion, so one has to consider a potential term [9, 10]. However, it is found in the aBD model including matter contribution that we have a de Sitter solution ($H =$ const.) for the FRW metric, given by

$$a = a_0 e^{Ht}, \quad \phi = \phi_0 e^{-3(1+\omega_m)Ht}, \quad \rho_m = \rho_0 a^{-3(1+\omega_m)} \tag{6.1}$$

with constants $a_0,\ \phi_0,\ \rho_0$ and $P_m = \omega_m \rho_m$. In particular, λ and ρ_0 satisfy the following relation:

$$\lambda = 2\eta + (1-\omega_m^2)\omega_1 + 1/3, \quad \rho_0 = -9\omega_m \phi_0 a_0^{3(1+\omega_m)} H^2(\eta + \omega_1(1+\omega_m)). \tag{6.2}$$

For $\rho_m = 0$ (vacuum) case, the above relation yields

$$\omega_1 = -2\eta + \lambda - 1/3. \tag{6.3}$$

It should be pointed out that in the BD limit, i.e., $\eta \to 1,\ \lambda \to 1$ the parameter $\omega_1 = \omega_{\rm BD}$ becomes a negative value, $\omega_{\rm BD} = -4/3$ which conflicts with the lower bound $\omega_{\rm BD} > 50000$, even if it satisfies the ghost-free condition of $\omega_{\rm BD} > -3/2$ [18]. However, in the aBD case this problem can be circumvented by choosing η to be $\eta > 4/3$ or $\eta < 0$ for the bound (5.15), obtained from substituting (6.3) into the ghost-free condition of Eq. (3.10) of Sec. 3, when $\eta_1 = \eta_2$. It is worth noticing that the allowed range is not in conflict with the special case of $\eta = 2$.

The second one is a speculation about quantum gravity. In Refs. [25–27], the one-loop counter terms in the pure BD theory is calculated and it is shown that the BD theory is not renormalizable unless curvature squared terms are included. Likewise, we suspect that adding curvature squared terms [12, 16, 28, 29] in the aBD theory might constitute the UV completion of the theory. The details are beyond the scope of the present paper, but the subject deserves further investigations.

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